EVOLUTION OF
PTYGYMATIC FOLDING

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Abstract: It is argued that ptygmic veins generally are caused by a component of compressive strain parallel to the vein. Pinch-and-swell and boudinage structures develop when the veins are so oriented that a component of extensive strain is parallel to the veins.

Analysis of strain geometry give relationships between angular attitude of veins (relative to the principal strain axes in rocks) and magnitude of stretching (pinch-and-swell), or magnitude of shortening (ptygmic folds).

When rocks are strained plastically there always are some directions along which compressive longitudinal strain occurs and others along which extensive longitudinal strain occurs. Ptygmic folding and pinching-and-swelling or boudinage are consequently complimentary phenomena which can develop simultaneously in response to the same stress system in the host rock.

Experiments with putty as host and plasticene sheets as veins give results identical to natural ptygmic and, pinch-and-swell structures.

The longitudinal compressive force which is responsible for buckling of veins is caused by drag or skin friction along veins due to plastic compressive
flow in less competent host rocks. An equation for this compressive force as a function of dimensions of vein and principal stresses in the host rock is developed.

The buckling of an enclosed vein is resisted by two distinct forces: 1) the strength (viscosity) of the vein itself, and 2) the strength (viscosity) of the enclosing rock. Resistance 1) is at a minimum when the entire vein makes one half-wave; resistance 2) is least when the vein makes an infinite number of infinitely small waves. The actually developed wave length (or rather length of arc) represents a compromise between these two tendencies. Equations for the two resistance forces and the most "stable" initial wave length are presented. This initial wave length, $\lambda_1$, of ptygmatic veins is the same as the length of arc of more mature folds. $\lambda_1$ is the wave length which requires minimum compressive force to develop.

It is implied that a study of geometry and dimensions of ptygmatic and pinch-and-swell veins in the field gives valuable information not only as to the geometric type of strain which has effected the considered rock complex, but also as to relative strength and creep viscosity of rocks.

**Introduction.**

When Sederholm in 1913 first coined the term ptygmatic folds for the sinuous shape of many veins in gneisses and migmatites, he considered the folding as a secondary phenomenon caused by shortening of plastic host rocks in directions parallel to the veins. This conclusion has in principle been accepted by several students of such veins (Erdmannsdörfer, 1938; Kuenen, 1938; Spurr, 1923; Godfrey, 1954; Ramberg, 1952–56). Other geologists, however, have suggested that ptygmatic structure develops without host-rock compression. Read, 1928, assumes that emplacement along irregular zig-zag fracture systems results in ptygmatic veins, and Niggli, 1925, Wilson, 1952, proposed that forcefull injection of magma in incompetent rocks gives rise to ptygmatic structures. The writer has had opportunity to study numerous examples on veins with ptygmatic structure in gneisses and schists, mostly pegmatites and aplites in high grade gneisses and quartz-calcite veins in low grade schists, and he has found no reason to doubt that this striking structure essentially is caused by a component of shortening in host rocks parallel to originally more or less planar veins. Most arguments against this view are probably due to lack of obvious strain features in many ptygmatic veins. This fact, however, only shows that the deformation takes place under somewhat different conditions (greater plasticity or slower strain rate) than those existing under evolution of rocks commonly accepted as
tectonites. Strain experiments with putty and plasticene subsequently to be discussed have brought convincing support to the strain theory of ptygmatic folds.

At the proper difference in competency between vein and host rock — the vein must be somewhat more competent than the host — the vein can only adjust to plastic compressive strain in the host rock by a folding mechanism, provided the vein is properly oriented in relations to the strain geometry of the host rock. In this connection one may note that Sederholm (1913 and 1926) seems not to have realized the necessity of the vein material being more competent (mechanically strong) than the host. In his first paper, in fact, he assumed that the vein material was liquid during the folding because of lack of cataclastic phenomena in the vein. In his later paper, however, Sederholm considered the folding to have occurred in the crystalline state, the lack of cataclasis being explained by simultaneous recrystallization.

Plastic compression along some directions in rock complexes must be associated with simultaneous extensions in directions that make large angles to the compression, provided that volume remains essentially constant during strain. Relatively competent sheets of rocks which make large angles with direction of maximum compression will therefore develop tension fractures or necked-down regions. One should consequently expect boudinage and pinch-and-swell structures to be associated with ptygmatic structures in the field as unseparably as extensive strain is associated with compressive strain in plastic deformation at constant volume. Kuenen, 1938 p. 23 notes such field association between stretching and ptygmatic folds. Field experience of the present writer supports this expectation: pinch-and-swell structure of quartz-feldspar veins or boudinage structure of other competent sheet-shaped rocks are generally so intimately associated with ptygmatic structures in the field that a complementary relationship of the kind mentioned above is strongly suggested, see Ramberg, 1956, p. 192.

Boudinage and pinch-and-swell structures as consequence of plastic extensive flow in heterogeneous rocks have been discussed in some detail in a previous paper, (Ramberg, 1955). In the present paper emphasis is put on plastic shortening in strained rocks and the consequent evolution of ptygmatic veins.
Ptygmatic folds and pinch-and-swell structures
in relation to geometry of strain.

Strain experiments with putty as incompetent host and plasticene as competent vein have resulted in structures strikingly similar to those seen in many veined schists and gneisses (migmatites), see pls. 1 to 6. The evolution of such structures is probably most rigorously explained in view of the general geometry of plastic strain. If the geometry of strain is studied in detail we find that many structural features, which have been considered inconsistent with the secondary-folding theory of ptygmatic veins (Wilson, 1952, p. 16), actually are to be expected when competent sheets are embedded at various angles in yielding incompetent matter. Such structural features supposed to be characteristic of primary injection of ptygmatic veins, also developed in the strain experiments performed by the writer, see p. 127.

To clarify the relations between host-rock structure and the structure of deformed enclosed veins, we shall consider the geometry of strain. For simplicity we shall assume essentially homogeneous strain in the incompetent host rock (or putty) except for the unavoidable boundary effects close to ptygmatically folded or pinching-and-swelling veins. Let us consider the following geometric classes of homogeneous plastic strain:

1) Pure shear (in two dimensions): Maximum compressive strain parallel to z-axis in a rectangular coordinate system; maximum extensive strain parallel to x-axis, and zero strain along y-axis. No rotation of coordinate axes during strain; axes for principal stress coincide with axes of principal strain.

2) Irrotational three dimensional homogeneous strain:
   a): Maximum compressive strain parallel to z-axis; extensive strain equal in all directions in the x, y-plane. (Uniaxial compression.)
   b): Maximum extensive strain parallel to z; compressive strain equal in all directions in the x, y-plane. (Uniaxial extension.)

3) Simple shear or rotational homogeneous strain. Shearing strain by movement parallel to x, y-plane in x-direction. Length of dimensions along z, x and y remain constant.
Fig. 1: Hyperbolic flow lines in a $x, z$ section of a body under pure-shear deformation. The traces of the two $45^\circ$ planes of no infinitesimal longitudinal strain are shown.

*Pure shear deformation.*

To make the strain homogeneous throughout the incompetent body, we must assume frictionless interfaces between the strained body and the medium which transmits stresses. This is not generally true in rocks, but the condition of homogeneous pure shear may be approached rather closely within a limited volume in the interior of a large homogeneous rock body.

Under pure strain the flow lines in the $x, z$-plane are families of rectangular hyperbolas with origin in the center of the strained body if we assume constant volume during strain, and disregard elastic effects, see fig. 1. During this kind of plastic strain, then, any given small particle in the body moves along a hyperbola of the form $x \cdot z = k$ where $x$ and $z$ are abscissa and ordinate respectively, and the constant $k$ equals the area of the rectangle limited by the coordinates $x$ and $z$ and the coordinates axes. The hyperbolic shape of the flow lines follows from the requirement of constant volume during strain, and constant dimension of the strained body parallel to $y$.

Since the strain is homogeneous the movements in the environ-
ment to any given particle relative to that particle are the same independent of the position of the reference particle. The relative motion of matter in the vicinity of any given particle follows hyperbo­
lic curves with center in the reference particle which itself of course is moving along a flow line. The relative flow is thus identical to the flow around the fixed origin of the coordinate system in the center of the plastic body. Any deformation which we now shall consider in relation to a coordinate system fixed in the center of the deformed body will consequently be identical to deformations in parallel direc­
tions anywhere throughout the strained body.

Since we are concerned with ptygmatic veins (plastic shortening of host rock parallel to vein), and pinch-and-swell structures of veins (stretching parallel to vein) housed in plastic rocks, it is of prime interest to determine the angular attitude of directions along which compressive — respective extensive — strain occurs under pure shear. We shall firstly consider directions parallel to the $x, z$-plane.

Let $x$ and $z$ be coordinates to a point $x, z$, and $l$ its distance from origin. We are seeking $dl/l$, i.e. the infinitesimal longitudinal strain along $l$, as a function of the angle between $l$ and the $z$ axis, and of the principal infinitesimal compressive strain in the body along $z$, $dz/z$.

According to fig. 1 the following relations are true:

1) \[ l^2 = x^2 + z^2 \]
2) \[ x \cdot z = k. \]

Eq. (2) in (1) gives:
3) \[ l^2 = \frac{k^2}{z^2} + z^2. \]

This can be differentiated:

\[ l \cdot dl = (z - k^2 z^{-3}) dz, \]
which divided by eq. (3) gives the wanted quantity:

4) \[ \frac{dl}{l} = \frac{1 - k^2 z^{-4}}{k^2 z^{-4} + 1} \frac{dz}{z}. \]
5) \[ \tan \alpha = \frac{x}{z}, \]

see fig. 1. Combining eq. (2) and (5) gives:
6) \[ \tan \alpha = k z^{-2}. \]
Accordingly, eq. (4) can be written as follows:

\[ \frac{dl}{l} = \frac{1 - \tan^2 a \, dz}{1 + \tan^2 a \, z} \]

The infinitesimal principal strain, \( \frac{d\zeta}{\zeta} \), is negative in pure shear inasmuch as shortening takes place parallel to \( z \). The sign of \( \frac{dl}{l} \) which shows whether \( l \) shall suffer shortening or lengthening, depends then only upon the angle \( a \) as follows:

\[ 45^\circ < a < 90^\circ \quad \text{gives} \quad \frac{dl}{l} > 0 \]

\[ a = 45^\circ \quad \text{gives} \quad \frac{dl}{l} = 0 \]

\[ 0 < a < 45^\circ \quad \text{gives} \quad \frac{dl}{l} < 0 \]

In words: at infinitesimal strain all directions making smaller angle than \( 45^\circ \) with \( z \) become shortened by relative amounts varying from \( \frac{dz}{z} \) at \( a = 0 \) to zero at \( a = 45^\circ \). Directions making larger angle than \( 45^\circ \) with \( z \) are lengthened by amounts varying from \( \frac{dx}{x} = -\frac{dz}{z} \) at \( a = 90^\circ \) to zero at \( a = 45^\circ \), see fig. 1.

For directions deviating from the \( x, z \)-plane we find that infinitesimal lengthening and shortening respectively are confined to angular directions limited by two planes parallel to \( y \) and making \( 45^\circ \) angle to \( z \) and \( x \). Shortening effects directions in the angular region bisected by the \( y, z \)-plane, lengthening effects directions within the region bisected by the \( y, x \)-plane. The two sets of planes of no longitudinal infinitesimal strain which bisect the angle between the \( x \) - and \( z \) axes coincide with the circular cross sections of the strain ellipsoid at infinitesimal distortion from a sphere. However the angular attitude of the planes of no longitudinal infinitesimal strain remains constant at \( 45^\circ \) inclination to \( z \) during the entire deformation whereas the circular cross sections in the strain ellipsoid of course rotate toward the \( x, y \)-plane.

During deformation matter moves at right angle across the planes of no infinitesimal longitudinal strain following the hyperbolic flow lines.

It is important for the following discussion to distinguish between imaginary mathematical planes or lines in the strained body, and
Fig. 2: Pure-shear type deformation of a sphere shown in one quadrant of the $x, z$-plane. $I_2$ is the trace of the circular cross section. The curve $x_1z_1 - b - x_2z_2$ is the hyperbolic trajectory of a particle existing at $x_1z_1$ prior to straining and ending up at $x_2z_2$ after finite strain.

real sheets or rods which are made up of the constitutent particles of the yielding body or of some incorporated foreign material (veins, etc.). The movement and strain of such sheets and rods — their rotation, their longitudinal and shearing strain, their displacement in relation to each other etc. — are controlled by the movement of particles along the hyperbolic flow lines. One must bear in mind that the rotation and strain of materialistic sheets often is quite different from the rotation and strain of mathematical planes which can be constructed in the strain ellipsoid. The $45^\circ$ planes of no infinitesimal longitudinal strain are imaginary mathematical sections through which matter flows. So are the circular cross sections in the strain ellipsoid; although these planes rotate during strain, properly oriented sheets of matter rotate faster and may sooner or later overtake the circular cross sections and rotate through them. The circular cross sections are planes of no finite longitudinal strains, i.e. $\frac{\Delta l}{l} = 0$, because any rod of matter from origin to the circumference of the circular cross section has the same length as this rod had in the original unstrained sphere. This
does not mean, though, that rods or sheets lying parallel to the circular cross sections have not suffered longitudinal strain. Indeed such bodies have first been shortened and then stretched exactly to their original length during their rotation toward the circular cross sections in the strain ellipsoid, see fig. 2.

It is noteworthy that at the moment sheets or rods of matter rotate through the circular cross sections, the longitudinal infinitesimal strain, \( \frac{dl}{l} \), as related to an infinitesimal angle of rotation, \( da \), has a finite positive value. Only at the very instant the rotating sheets or rods are parallel to the 45° oblique planes of no longitudinal infinitesimal strain is \( \frac{dl}{l} = 0 \).

The above-mentioned difference between the circular cross sections in which finite longitudinal strain is zero and the 45° planes of no longitudinal infinitesimal strain is of consequence in the study of ptygmatic folding and pinch-and-swell structures.

The rotation of the circular cross section as a function of finite principal strain, \( \frac{Az}{z} \), follows from fig. 2.

We have:

\[
\frac{z_2 - z_1}{z_1} = \frac{Az}{z} = \frac{l_2 \sin \varphi_2 - l_1 \sin \varphi_1}{l_1 \sin \varphi_1}
\]

where subscript 1 refers to original length and angle, subscript 2 to strained length and angle. Since we seek the angle of the circular cross section after finite strain, \( l_2 \) must be equal to \( l_1 \), and eq. (8) becomes:

\[
\frac{Az}{z} = \frac{\sin \varphi_2}{\sin \varphi_1} - 1.
\]

\( l_1 \) and \( l_2 \) are symmetric in relation to the 45° plane of no infinitesimal strain, hence \( \varphi_2 = \alpha_1 \) or \( \varphi_1 + \varphi_2 = 90° \) from which follows: \( \sin \varphi_1 = \cos \varphi_2 \). Inserting this in eq. (9) gives

\[
\tan \varphi_2 = \frac{Az}{z} + 1,
\]

where \( \varphi_2 \) is the angle between the \( x, y \)-plane and the circular cross section, and \( \frac{Az}{z} \) is finite principal strain in \( z \) direction.
It is of interest for our purpose to determine interrelations between angular attitude and angular rotation of embedded sheets (veins), longitudinal strain in various directions in such sheets, and strain of the whole incompetent rock body (putty) in the direction of principal strain axis.

Longitudinal strain in three sheets parallel to the coordinate planes is trivial. Sheets parallel to the $x$, $z$-plane are shortened by an amount $\frac{\Delta z}{z}$ parallel to $z$, and lengthened by an amount $\frac{\Delta x}{x}$, parallel to $x$. $\frac{\Delta z}{z}$ is finite compressive strain parallel to $z$, and $\frac{\Delta x}{x}$ finite extensive strain parallel to $x$.

If the enclosed sheet is identical to the surrounding body in mechanical properties or more soft, neither ptygmatic folding nor pinch-and-swell will develop as was verified by several tests. Sheets more competent than the host adjust to shortening in $z$ direction by ptygmatic folding about an axis more or less parallel to $x$. In attempt to adjust to the extensive strain parallel to $x$ the same sheet may develop pinch-and-swell (or boudinage) structure with necked down
Fig. 4: Pinch-and-swell formed in a vein which was folded when oriented in the compression segment at the early stage of deformation, and stretched when rotated into the extension segment. The geometry of strain is pure-shear type.

zones parallel to the $x, y$-plane. Fig. 3 shows a block diagram of a vein deformed in this manner. It is worth noting at this point that pinch-and-swell (or boudinage) may develop by stretching along $x$ in veins which cut across schistosity at a large angle. The kind of schistosity which is caused by strain in crystalline rocks is here assumed to develop perpendicular to maximum compressive finite strain, i.e. parallel to the $x, y$-plane in pure strain, rather than parallel to maximum shear strain.

Sheets parallel to the $y, z$-plane are shortened by an amount $\Delta z/z$ in the $z$ direction and remain unstrained parallel to $y$. Ptygmatic folding about axis parallel to $y$ is hence possible for competent veins oriented parallel to the $y, z$-plane in incompetent hosts, but pinch-and-swell structure cannot develop in sheets with this orientation.

Sheets parallel to the $y, x$-plane will be stretched in $x$ direction and unstrained in $y$. Such sheets may thus develop boudinage or pinch-and-swell by fractures or necked down regions parallel to $y$, fig. D, pl. 2. Ptygmatic folding is not possible.
The behavior of sheets inclined to some or all three co-ordinate axes is more interesting. Sheets parallel to $y$ but oblique to $z$ and $x$ are effected by rotation and shortening or lengthening in the $x$, $y$-direction depending upon original and final angular attitude. Let $a_1$ and $a_2$ be original and final angles respectively, between $z$ and the sheet. At $a_1 < 45^\circ$ shortening in the $x$, $z$-direction occurs until rotation reaches the $45^\circ$ plane of no infinitesimal strain. In this period of deformation, then, ptygmatic folding may develop with axis parallel to $y$. Continued compressive strain along $z$, however, makes the ptygmatically folded sheet rotate past the $45^\circ$ angle into the region of lengthening. As a result the previously formed folds may either be stretched out, as in some of the experiments, or the folded sheet may disrupt into a row of sigmoidal boudins (see fig. 4). Incidentally, if ptygmatic folding and subsequent stretching which result from strain and rotation are reversible phenomena there are some simple relationships between the original angular attitude of an unstrained vein (sheet), and the angle at which the ptygmatic folds have just been straightened to restore the original length. At the moment of complete straightening of the vein, it is parallel to the circular cross section in a strain ellipsoid formed from a sphere which was undeformed at the moment compressive strain and ptygmatic folding of the vein started. Under conditions of pure shear we know that sheets of matter being parallel to the circular cross sections have rotated through an angle twice the angular difference between the circular cross section and the $45^\circ$ planes of no infinitesimal strain, see fig. 2. Hence if the angle between a straightened out vein and the plane of schistosity is $\varphi_2$ then the angle between the original underformed vein and the plane of schistosity was:

$$\varphi_1 = 90 - \varphi_2.$$  

There also exists a simple relation between finite principal strain parallel to $z$, $\frac{\Delta z}{z}$, and the angular attitude of a vein at the moment it is straightened out to its original length. At this stage the vein lies parallel to the circular cross section in the strain ellipsoid, and eq. (10) p. 107. is valid:

$$\frac{\Delta z}{z} = \tan \varphi_2 - 1.$$
Fig. 5: Pinch-and-swell formed in a vein lying in the extension segment but yet inclined to the principal extension axis, $x$. Pure-shear type deformation.

where $\varphi_2$ is the angle between the straightened out vein and the plane of schistosity.

Eqs. (11) and (12) above are only rigorously applicable to pure shear strain in rocks if the deformations in the vein — its folding and subsequent straightening — are reversible and there is no slip between vein and adjacent rock. Such reversibility is probably not approached very closely in rocks. Some of the experiments, however, showed considerable reversibility of this kind, see for example fig. G, pl. 4.

Sheets parallel to $y$, but originally making greater than $45^\circ$ angle with the $z$ axis are effected only by stretching in the $x, z$-direction and may consequently develope pinch-and-swell or boudinage structure. It is interesting to realize that these structures thus can form in competent bodies with their long axis inclined to the schistosity in the host rock, see fig. 5.

Finite longitudinal strain of a sheet in the $x, z$-direction, $\frac{\Delta l}{l}$, is related to the angle of rotation, $\Delta \alpha = \alpha_2 - \alpha_1$ which in turn is related to the principal compressive strain of the whole plastic body,
These relationships may be useful as structural tools for if amount of stretching and shortening of a folded or pinching-and-swelling vein can be measured in the field, as well as its angle with the host-rock schistosity, the quantities $\Delta l/l$ and $a_2$ are known. $a_1$ and principal compressive strain along $z$ in the host rock can then be estimated from the equations. (It is assumed here that the schistosity is produced by the same strain as that which produces ptygmic folds and pinch-and-swell structures. If the observed schistosity is result of pre-vein deformation or sedimentary laying, or if it is parallel to shear strain rather than perpendicular to compressive strain, the following calculations do not hold.)

According to fig. 1 we can write:

13) \[ l = \frac{z}{\cos a}, \]

hence:

14) \[ \frac{\Delta l}{l} = \frac{l_2 - l_1}{l_1} = \frac{z_2 \cos a_1}{z_1 \cos a_2} - 1 \]

Now $x \cdot z = k$, and $\tan a = \frac{x}{z}$, therefore:

15) \[ z = \sqrt{\frac{k}{\tan a}}, \]

which can be inserted in eq. (14):

16) \[ \frac{\Delta l}{l} = \sqrt{\frac{\tan a_1 \cos a_1}{\tan a_2 \cos a_2} - 1} = \sqrt{\frac{\sin a_1 \cos a_1}{\sin a_2 \cos a_2} - 1} \]

The angle $a_1$ can be found from eq. (16) if $\frac{\Delta l}{l}$ and $a_2$ are measured on ptygmic — or pinch-and-swell veins in schists or gneisses. The principal compressive strain, $\frac{\Delta z}{z}$, can be estimated from the relation:

17) \[ \frac{\Delta z}{z} = \sqrt{\frac{\tan a_1}{\tan a_2} - 1}, \]

which follows from eq. (15).

As an example on amount of longitudinal strain, $\frac{\Delta l}{l}$, caused by rotation of a vein from $a_1$ to $a_2$, we shall assume $a_1 = 20^\circ$ and $a_2 =$
Fig. 6: Deformation of pure-shear type shown in one quadrant of a $y, z$ section. Compression from $z_1$ to $z_2$ has caused shortening of line $l$ from $l_1$ to $l_2$ and rotation from $\beta_1$ to $\beta_2$.

45°. Inserting these values in eq. (16) gives $\frac{\Delta l}{l} = -0.1925$ or 19.25 % shortening. Rotation from $\alpha_1 = 45^\circ$ to $\alpha_2 = 80^\circ$ results in $\frac{\Delta l}{l}$ 0.718, or 71.8 % lengthening.

During pure shear a sheet parallel to $x$ but inclined to $z$ and $y$ will be stretched parallel to $x$ by an amount $\frac{\Delta x}{x}$ and can consequently develop boudinage or pinch-and-swell structure with fractures or necked-down zones parallel to the $y, z$-plane. The $y, z$-direction in the sheet becomes shortened and rotates around the $x$ axis which also is the axis of ptygmatic folds in the sheet. The amount of shortening in the $y, z$-direction, $\frac{\Delta l}{l}$, depends upon the angle, $\beta$, which the sheet makes with the $z$-axis before and after straining.

According to fig. 6 the following relation is valid:

18) \[ l = \frac{y}{\sin \beta} \]

where $y$ is constant because in pure shear the motion of any given particle is parallel to the $x, z$-plane. The relative shortening in the $y, z$-direction in a vein is then:

19) \[ \frac{l_2 - l_1}{l_1} = \frac{\Delta l}{l} = \frac{\sin \beta_1}{\sin \beta_2} - 1 \]

According to these equations one may for example estimate the principal strain in $z$ of the host rock if $\frac{\Delta l}{l}$ and $\beta_2$ are measured on ptygmatic veins in the field. From fig. 6 follows equation
Fig. 7: Deformation of pure-shear type shown in one quadrant of the $x, y$-plane. Extension from $x_1$ to $x_2$ has caused elongation of line $l$ from $l_1$ to $l_2$ and rotation from $y_1$ to $y_2$.

\[ \frac{\Delta z}{z} = \frac{z_2 - z_1}{z_1} = \frac{\cot \beta_2}{\cot \beta_1} - 1 \]

where $\beta_2$ is directly measurable and $\beta_1$ may be calculated from measured elongation, $\frac{\Delta l}{l}$, according to eq. (19).

Sheets parallel to the $z$ axis but inclined to $x$ and $y$ are effected by principal compressive rock-strain parallel to $z$, and by various amount of stretching in $x, y$-direction depending upon the angle $\gamma$ between the sheet and the $x$ axis. Consider the strain in the $x, y$-plane in which the projected flow lines are parallel to the $x$ axis, see fig. 7. According to this figure equation

\[ l = \frac{y}{\sin \gamma} \]

is valid, where $y$ is constant on account of the flow lines being parallel to the $x, z$-plane. Consequently the longitudinal strain is:

\[ \frac{l_2 - l_1}{l_1} = \frac{\sin \gamma_1}{\sin \gamma_2} - 1 \]

Measurement of elongation, $\frac{\Delta l}{l}$, (from separation of boudins or stretching of pinch-and-swell veins), and $\gamma_2$ enables one to calculate $\gamma_1$ by means of eq. (22). This is sufficient information, then, to estimate the principal extensive strain along $x$ in the host-rock body according to equation:

\[ \frac{\Delta x}{x} = \frac{\tan \gamma_1}{\tan \gamma_2} - 1. \]

As the last example on sheet orientation under pure shear we shall consider sheets inclined to all three coordinates axes. It is convenient to let the sheet go through origin of the coordinate system. Any
oblique sheet through origin must intersect the three axial planes. Let
us call the lines of intersection with the $x$, $y$-plane, the $x$, $z$-plane,
and the $y$, $z$ plane $l_{xy}$, $l_{xz}$, and $l_{yz}$ respectively. The oblique plane is
determined if two of the intersection lines are known. During pure
shear of the host rock (putty) an embedded oblique sheet will both
rotate and become effected by longitudinal strain. Rotation of the
oblique sheet can be studied in terms of rotation of its lines of inter­
section with the axial planes. Relationships between principle rock
strain in $z$ or $x$, and increments in the angles $a$, $\beta$, $\gamma$ are identical to
those discussed above in connection with the three sets of sheets
parallel to the $y$-, $x$- and $z$-axes respectively.

Fig. 8a: Block diagram of pure-shear effecting an oblique competent sheet of
matter embedded in incompetent material. The directions of maximum com­
pressive — and maximum extensive-strain in the sheet are shown. The directions
of maximum shear and zero longitudinal strain are also indicated. These directions
are the intersections between the oblique sheet and the two 45° planes of no
longitudinal strain (not shown here).
Any oblique sheet through origin must intersect the two $45^\circ$ planes of no infinitesimal longitudinal strain. The angle between the two lines of intersection is generally oblique. These are directions of maximum shear and no infinitesimal longitudinal strain in the sheet. The angles between the two directions of no infinitesimal longitudinal strain in the oblique sheet are bisected by the directions of maximum positive and maximum negative longitudinal strain respectively. The direction of maximum positive longitudinal strain in the sheet lies closest to, but is not parallel to, the $y, x$-plane, and the direction of maximum negative longitudinal strain in the sheet lies closest to the $z, y$-plane, see fig. 8a, b. The intersections between the circular cross sections in the strain ellipsoid and the oblique sheet are lines of no finite longitudinal strain in that sheet.

In general, then, an embedded competent sheet oblique to all the principal strain axes is compressed into ptygmatic folds in the direction of maximum compressive longitudinal strain in that sheet and
stretched into boudinage or pinch-and-swell structure in the direction of maximum extensive longitudinal strain. The axis of ptygmatic folds and the zones of necking down or boudin fractures of such oblique veins do in general make oblique angles with the host-rock schistosity and lineation. This conclusion is of significance in the argument as to whether ptygmatic folds are tectonic features or produced by primary injection.

Compressive strain parallel to z, uniform extensive strain in the x, y-plane.

The movement of any given particle in the yielding body follows a flowline lying in a radial plane which contains the z-axis. The equation for the flowline in the radial plane is:

23) \( r^2 = k \) where \( r = \sqrt{x^2 + y^2} \) is the distance from z-axis.

Flowlines in the x, z- and y, z-planes respectively, are given by eq. (24):

24) a) \( x^2 z = k \),
    b) \( y^2 z = k \).

Eqs. (23) and (24) follow from the condition of constant volume during plastic strain.

The family of flow surfaces are hyperbolic revolution surfaces rotated about the z-axis. We are interested in finding how longitudinal strain in an embedded sheet varies with its angular deviation from the z axis, and with the principal compressive strain in z. Because of rotational symmetry around z, all sections parallel to z give identical pictures with respect to the plastic strain. Let x be the abscissa, and z the ordinate in such a section. The distance from origin to a point \( x, z \), is (fig. 9).

25) \( l^2 = x^2 + z^2 \).

We are concerned about the infinitesimal longitudinal strain, \( \frac{dl}{l} \), experienced by a sheet of matter making an angle, \( \alpha \), with z when it rotates through \( da \), and the infinitesimal principal strain along z is \( \frac{dz}{z} \).
Eq. (24a) in eq. (25) gives:

\[ l^2 = \frac{k}{z} + z^2. \]  

Differentiating eq. (26) and dividing by itself results in:

\[ \frac{dl}{l} = \frac{1 - \frac{1}{2} k z^{-3}}{1 + \frac{1}{2} k z^{-3}} \cdot \frac{dz}{z}. \]  

\[ \tan \alpha = \frac{x}{z} = \sqrt{k z^{-3}}, \]  

according to eq. (24). Consequently eq. (27) can be written:

\[ \frac{dl}{l} = \frac{1 - \frac{1}{2} \tan^2 \alpha}{1 + \frac{1}{2} \tan^2 \alpha} \cdot \frac{dz}{z}. \]

Since the principal strain along \( z \) is compressive, \( \frac{dz}{z} \) is negative and \( dl/l \) is zero when \( 1 - \frac{1}{2} \tan^2 \alpha = 0 \). I.e. \( \tan \alpha = \sqrt{2} \), and \( \alpha = 54^\circ 44' \). Thus for linear compression along \( z \) with uniform extension in all directions in planes perpendicular to \( z \) the directions with zero longitudinal infinitesimal strain lie on a double-cone surface with half angle equal to \( 54^\circ 44' \), and axis of revolution coinciding with
the $z$ axis. Directions with $\alpha < 54^\circ 44'$ have negative $dl/l$ and suffer shortening; directions with $\alpha > 54^\circ 44'$ have positive $dl/l$ and become stretched.

It is of interest to determine relations between angular rotation away from the $z$ axis of a sheet of matter, $\Delta \alpha = \alpha_2 - \alpha_1$, and finite principal compressive strain in the host rock, $\Delta z/z$. According to eq. (28) $z_1$ and $z_2$ are related to $\alpha$ as follows:

\begin{align}
30) \quad \tan \alpha_2 - \tan \alpha_1 &= \sqrt[3]{z_2^{-3} - z_1^{-3}} / \sqrt[3]{z_1^{-3}}
\end{align}

\begin{align}
31) \quad \frac{\tan \alpha_2}{\tan \alpha_1} &= \sqrt[3]{\frac{z_1^3}{z_2^3}}
\end{align}

Incidentally, the analogous relation for pure shear is:

\begin{align}
32) \quad \frac{\tan \alpha_2}{\tan \alpha_1} &= \frac{z_1^2}{z_2^2}
\end{align}

from eq. (15) p. 112 showing that rotation of inclined sheets is somewhat larger for given principal compressive strain in pure shear than in uniaxial compression with rotational symmetry.

Finite longitudinal strain in the inclined sheet, $\Delta l/l$, as a function of rotation from $\alpha_1$ to $\alpha_2$ due to compression along $z$ can be shown to be:

\begin{align}
33) \quad \frac{\Delta l}{l} &= \sqrt[3]{\frac{\tan^2 \alpha_1 \cos \alpha_1}{\tan^2 \alpha_2 \cos \alpha_2}} - 1
\end{align}

(Compare the corresponding equation for pure shear, eq. (16) p. 112).

Let us now consider the behavior of competent sheets with various angular attitudes to the principal compressive strain axis. Sheets parallel to $z$ will behave identically independent of their inclination to the $x$ and $y$ axes because of rotational symmetry of strain. Such sheets are shortened along $z$ by an amount equal to the principal compressive strain in the host, and stretched in directions parallel to the $x$, $y$-plane. Ptygmatic folding and pinch-and-swell structures in corresponding directions are accordingly to be expected in analogy to sheets parallel to the $x$, $z$-plane in pure shear. However, in the present case the extensive strain in the $x$, $y$-plane is numerically smaller than the compressive strain in $z$. 
Sheets perpendicular to \( z \) are stretched uniformly in all directions during strain thus giving rise to symmetric two dimensional boudinage or pinch-and-swell structures such as that shown in fig. C, pl. 6 in Ramberg, 1955.

All sheets inclined to \( z \) are stretched in directions parallel to the \( x, y \)-plane and may consequently develop boudinage and/or pinch-and-swell structures in that direction. As to whether an oblique sheet should become compressed or stretched in directions at right angle to its intersection with the \( x, y \)-plane depends upon the angle, \( \alpha \), between the sheet and \( z \)-axis. If \( \alpha > 54^\circ 44' \), the said direction suffers stretching, and sheets with this inclination may develop two-dimensional pinch-and-swell structures. If \( \alpha < 54^\circ 44' \) the same direction in the sheet is effected by shortening with consequent formation of ptygmatic folds.

The intersections between an oblique sheet and the cone-shaped surface of zero longitudinal infinitesimal strain correspond to directions in the sheet along which infinitesimal longitudinal strain vanishes.

\[ \text{Equal compression in all directions in } x, y \text{-plane, extension in } z. \]

In a \( x, z \)- (or \( y, z \))-plane the flow lines in this case also follow hyperbola of the character: \( x^2 z = k \) (or \( y^2 z = k \)), and the flow surfaces are hyperbolic revolution surfaces with \( z \) as axis of rotation.

In order to determine the angular attitude of directions of no infinitesimal longitudinal strain as well as regions for compressive — and extensive strain respectively, we proceed as in the previous situation see eqs. (25) to (29) p. 118). If \( \alpha \) is the angle between \( z \) and a given direction we find that infinitesimal longitudinal strain along this direction vanishes at \( \alpha = 54^\circ 44' \). At \( \alpha > 54^\circ 45' \) strain is compressive, i.e. \( \frac{dl}{l} < 0 \); at \( \alpha < 54^\circ 44' \) strain is extensive i.e. \( \frac{dl}{l} > 0 \). The cone of zero infinitesimal longitudinal strain is thus the same as in uniaxial compression, but the regions of compression and extension respectively, have been reversed as compared with the previous case.

In analogy with the case of compression along \( z \), it is simple to predict how embedded competent sheets with various angular attitude should behave under this type of plastic strain. A new situation, however, occurs when sheets are more or less parallel to the \( x, y \)-
Fig. 10: A $x, y$ section through a ptygmatic vein deformed in uniaxial extension type strain. The vein was originally a planar body oriented approximately parallel to the $x, y$ plane.

plane which is effected by compressive strain in all directions. Such sheets must buckle in an intricate manner in attempt to adjust to shortening in all directions. Fig. B, pl. 6 shows experimental results of this kind of deformation. It is impossible to assign a fold axis to such crumpled veins, even if the host rock have developed a distinct fold axis and elongation during the same period of strain.

It is interesting to note that cuts across $z$ through such complexly folded ptygmatic structures may show contorted veins with closed paths, see fig. 10. Cuts parallel to $z$, however, will reveal rather ordinary ptygmatic pattern. Under two dimensional compression two mutually perpendicular sheets both parallel to $z$ will be ptygmatically folded simultaneously. Such features are not uncommon in veined gneisses and migmatites.

**Simple shear deformation.**

Let the movement be parallel to the $x, y$-plane and in the $x$ direction. The two sets of planes in which longitudinal infinitesimal strain vanises, are parallel to the $x, y$-plane and the $y, z$-plane respectively. The direction of maximum infinitesimal extensive strain is parallel to the $x, z$ plane and makes $45^\circ$ angle with $x$; the direction of maximum infinitesimal compressive strain is parallel to the same plane and makes $90^\circ$ angle with maximum extensive strain. Conse-
Fig. 11: Some geometric relationships in simple shear deformation.

sequently all directions in the first and third quadrants are effected by infinitisimal extensive strain (positive $\frac{dl}{l}$), and directions in the second and fourth quadrant are effected by infinitisimal compressive strain, (negative $\frac{dl}{l}$).

The amount of finite longitudinal stretching or shortening of sheets as a function of their angular rotation during finite strain is readily found, see fig. 11:

$$l = \frac{z}{\cos \alpha}, \quad \frac{\Delta l}{l} = \frac{\cos \alpha_1 - 1}{\cos \alpha_2},$$

where $z$ is constant during simple shear.

$\Delta l/l$ is positive when $\alpha_2 > \alpha_1$ (only acute angles are considered). This can only happen either when the sheet originally is in the first and third quadrants or when a sheet in the second or fourth quadrants is rotated into the first and third quadrants through and angle of rotation $\Delta \alpha > 2\alpha_1$. If a sheet in the second quadrant is rotated less than $2\alpha_1$, then $\alpha_2 < \alpha_1$ and consequently $\frac{\Delta l}{l}$ is negative. The finite longitudinal strain caused by rotation from second to first quadrant is the result of a compressive strain (until $\alpha_2 = 0$) followed by an extensive strain. If $|\alpha_2| = |\alpha_1|$ or $\Delta \alpha = 2\alpha_1$, the compressive strain is exactly matched by the subsequent stretching and $\Delta l/l = 0$.

Whereas the one plane of no finite longitudinal strain rotates during deformation, both planes of no infinitisimal strain remain...
fixed parallel to the $x$, $y$- and $y$, $z$-planes during the entire deformation.

Let us now consider the theoretically expected behavior under simple shear of competent sheets embedded at various orientations relative to the coordinate axes. Competent sheets parallel to the $x$, $y$-plane should remain parallel to this plane. Since $\frac{dl}{l} = 0$ as well as $\Delta l/l = 0$ in this plane such sheets should neither suffer stretching nor folding, as was verified by experiments, figs. A, B, C, pl. 1.

Sheets parallel to $x$, $z$ should be compressed and thrown into folds along the direction of maximum compressive strain, and stretched to pinch-and-swell or boudinage in the direction of maximum extension strain.

Sheets parallel to $y$ but inclined to $x$ and $z$ are rotated and compressed to ptygmatic folds and/or stretched depending upon their original attitude and amount of shear movement, (pls. 1 to 6). The expected strain patterns of sheets inclined to all axes should be similar to those of sheets with corresponding attitude under pure strain as discussed above, fig. D, pl. 6.

Several experimental results of simple shear deformation are discussed below.

**Experiments.**

Putty was found to be a suitable imitation material for the incompetent host rocks, and plasticene, which is mechanically somewhat stronger than the putty, was used to simulate the veins. Incidentally, the same materials were used in the boudinage experiments of the writer (Ramberg, 1955). As to be expected, neither ptygmatic folding nor pinch-and-swell or boudinage developed under homogeneous strain when the «vein» material was identical to the host with respect to rheological properties or if the vein was softer as, for example, when colored putty was embedded in gray putty. Such embedded sheets remained straight during deformation. Longitudinal strain caused only thickening or thinning of the sheets. Of course, if deformation of the whole sample is non-homogeneous, as it tends to be close to boundaries, straight elongate inclusions of colored putty in grey putty would curve in various ways. This kind of deformation is outside the scope of this paper.
It was not practicable to make strain tests of all the possible orientations sheets discussed in the sections above, but sufficient number of runs were made to show that ptygmatic folding and pinch-and-swell or boudinage developed in excellent agreement with theoretical expectations.

*Simple shear experiments.*

It is relatively easy to maintain conditions of homogeneous strain in simple shear experiments. For this reason most experimental investigations of deformation of embedded sheets were made with this strain geometry. This fact, however, does not imply that the writer considers simple shear as the most common type of rock deformation. In his view the most frequent geometry of rock strain is one which contains an element of rotation but with all three principal strains unlike in magnitude and different from zero. Although the strain is rarely homogeneous over a large volume of rock complexes, it will essentially be so if the considered volume is small enough.

Two wooden blocks with a body of putty between were slid in shear movement relative to each other. The distance between the blocks was kept constant by placing a thin board of desired width between the blocks underneath the putty. The strain was homogeneous except in a narrow zone close to the wooden blocks and in the vicinity of the deformed sheets of plasticene. Because of friction between putty and the bottom board it is probable that the strain was non-homogeneous in the lower part of the putty body, but this did not effect the plasticene sheets or strips which were embedded in the upper part of the putty close to its free surface. In agreement with the theoretical models discussed in the previous section, the $y$-axis of the reference coordinate system is perpendicular to the free putty surface, the $x$-axis is parallel to the sliding movement, and the $z$-axis is perpendicular to the interfaces between putty and the wooden blocks. The plasticene sheets or strips, which were straight and uniformly thick, were embedded in the putty at various angles to the coordinate axes. Most sheets were so embedded that they did not touch the wooden blocks. Stress was therefore not transmitted to the ends of the plasticene sheets directly from the wooden blocks, but by frictional drag along the surfaces of the sheets caused by yielding putty. This is probably the most common situation in rocks also.
Sheets embedded parallel to the \( x, y \)-plane remain straight and unstretched during strain after even very large shear angle, see figs. A, B, C, pl. 1. This is to be expected in view of the fact that the \( x, y \)-plane is a plane of maximum shear strain and zero longitudinal strain.

Incidentally, the experimental fact that \( x, y \)-layers remain straight in simple shear suggests that dragfolds in layered rocks are not produced by shearing alone parallel to the folded layer; a component of compressive strain in direction along the layer seems imperative. Such longitudinal compressive strain could well be caused by plastic squeeze and consequent lateral flow of the adjacent incompetent layers. Simultaneous shear parallel to layering would make the folds tilt and produce the typical drag-fold geometry.

Sheets parallel to \( y \) and oriented in the first and third quadrants relative to \( x \) and \( z \) become folded with axes parallel to \( y \) as predicted in the theoretical section. [Note that the shear is performed by sliding the upper wooden block (which cuts the positive \( z \)-axis) in the direction of the negative \( x \)-axis. In the theoretical consideration of simple shear the movement was in opposite sense.] The axial plane of each fold may or may not be parallel to the long axis of the strain ellipsoid. In rocks this would mean that the axial plane of ptygmatic folds needs not be parallel to schistosity. Unless the sheets are parallel to the direction of principal compressive strain there is a component of shear strain parallel to the sheets; this shear tends to rotate each individual fold. The rotation is particularly evident in layers which make a small angle with the \( x, y \)-plane as shown in fig. C, pl. 6. Such ptygmatic folds have the characteristic shape of drag folds.

Wave length or rather arc length and amplitude are related to
thickness of the plasticene sheets as shown in figs. B, C, D, pl. 3,
and E, F, G, D, pl. 4.

Fig. 12 shows five steps in a sequence of deformation of three
sheets originally parallel to \( y \) but inclined at various angles to the
\( x \)-axis. It is seen how shortening as well as intensity of folding depends
upon angular attitude in a fashion expected theoretically. However,
it is noteworthy that the magnitude of shortening or lengthening is
somewhat less than geometrically ideal for some of the sheets. Thus,
for example, the stretching of the steepest sheet, which in the final
step of strain fractured to two boudins, is considerably less than pre­
dicted by the simple shear geometry. Likewise, the shortening of
the least steep sheet is smaller than geometrically ideal. The chief
reason for this is that the longitudinal strain in the putty adjacent
to the competent sheets is smaller in magnitude than it would have
been in this site if the relatively competent plasticene sheets were
absent. The pattern of striation on the putty surface as well as de­
formed circular marks close to folded sheets demonstrate the in­
hibiting effect which the competent sheets have on the compressive
strain in the adjacent putty, see figs. B, pl. 5, and B–F pls. 3 and 4.

Sheets parallel to \( y \) and inclined against \( x \) or \( z \) in second and
fourth quadrants were always lengthened, often without break but
sometimes producing pinch-and-swell or boudinage structures. Figs.
A, B, C. pl. 2, and fig. 12 show sheets with this orientation. The two
steepest sheets in fig. 12 have been rotated from the compression
quadrant to the extension quadrant. It is interesting to note how
the sheets firstly buckle into folds, then become stretched out, and
ultimately fracture to boudins. Actually, the middle sheet, which is
not broken in the figure, developed a fracture close to its middle when
strained beyond stage 5. For sake of clarity this state is not shown
in the figure.

A few tests with embedded plasticene sheets parallel to the free
surface of the putty (parallel to the \( x, z \)-plane) were performed. The
sheets developed folds essentially parallel to the long axis in the strain
ellipsoid.

Fig. D, pl. 6 shows the result of simple shear on two sheets em­
bedded at oblique angle to all three coordinate axes. The putty above
the sheets has partly been removed to clear the folded surface of the
sheets. Sheets with such oblique orientation do of course not have
parallel fold axes in spite of simultaneous deformation in response to the same homogeneous strain in the host rock. The geometry of strain performed in this experiment would correspond to the strain on the flank of a major fold with axis parallel to \( y (\gamma = b) \). Hence neither of the fold axes of the ptygmatically deformed sheets (veins) would be parallel to the major fold axis in the complex. Structures of this kind in rocks may at first glance erroneously be considered indicative of several periods of deformation.

Details of the nonhomogeneous strain adjacent to the ptygmatic folds are revealed by distortion of original circles impressed on the putty surface across the sheets and close to them. The crucial features of the pattern of this contact strain are the same as those of the contact strain adjacent to many natural ptygmatic veins such as that shown in fig. 13. It is interesting that this kind of contact structure of ptygmatic veins has been considered by G. Wilson (1952, p. 16) as evidence of primary folding caused by forceful injection of a very viscore melt in a less viscore host rock. We see here that such contact structures develop by secondary strain.

It is significant that the nonhomogeneous contact strain is confined to a rather narrow zone along the ptygmatic sheets. That means that the foldings of neighbor veins do not influence each other unless they are less than say one wave length apart. If ptygmatic folds for example were shear folds due to shearing fore and back along the schistosity plane of the host rock there would be a close parallelism between folded veins over a sizable portion of a given schist or gneiss.
Wave length and amplitude should be the same independent of individual vein thickness. This is not what one finds in nature.

Some observations of particular significance for the mechanism of folding as discussed in the following section shall now be mentioned.

If somewhat more than usual care was taken to produce plasticene strips with even thickness throughout their entire length at the same time as the enclosing putty was as homogeneous and free of lumps as possible, it was found that the resulting ptygmatic folds were very uniform in size and shape. Example of this is seen in pls. 3, 4. Most irregularities in the folds as found in the photos are due to heterogeneities in the putty (which often contained remnants of plasticene from previous tests), irregular thickness of the plasticene strips, and/or non-uniform strain due to various boundary effects.

If the ends of the plasticene strips were freely floating in homogeneous putty rather than touching the wooden blocks or some erratic inclusion in the putty, folding was always found to start along a central segment of the strip leaving the end portion seemingly straight and unaffected. In case several waves were produced along the active central segment, it was noted that they appeared simultaneously in the form of equal sized small-amplitude folds whose amplitude gradually increased. The wave length of course decreased correspondingly, but the length of arc along a fold remained constant (see also p. 148). Only in response to increased force applied in the shear motion of the wooden blocks would new folds outside the initial central active segment.

It was noted that the straight terminal segment unaffected by initial folding increased in length with increasing thickness of the strip; see for example fig. D, pl. 4. Of course, if the strip was thick enough it would not fold at all implying that the active central segment has shrunk to zero and the whole strip is occupied by the inactive straight end segments.

The ptygmatic folds could easily be unfolded again by reversing the direction of sliding of the wooden block as demonstrated in fig. G, pl. 4.

In addition to the various types of distortion of embedded competent sheets, other features of consequence for strain structures in rocks were revealed during the experiments with simple shear. A set of quite uniform shear fractures developed in the putty at about
\(20^\circ-25^\circ\) angle with the \(x, y\)-plane. The acute angle opened in the directions of shear motion, Pls. 1, 2, 4. Now maximum finite shear strain is parallel to the two circular cross-sections of the strain ellipsoid, one being fixed in the \(x, y\)-plane, the other rotating from the \(y, z\)-plane into the second and fourth quadrants. The experimentally produced fractures deviated considerably from either of these directions. The same holds for some fractures developed in shear experiments by Riedel, 1929. The orientation of the fractures is such that a component of extensive strain occurs perpendicular to the fractures. If deformation is continued after the fractures first appear on the putty surface, one finds that many of them open up a little at the same time as lateral shear displacement is evident. Thus the actual fractures are hybrids between “ideal” shear fractures which have no transversal extensive or compressive strain and “ideal” tension fractures along which shear strain vanishes.

On the surface of the strained putty a very conspicuous set of striations developed parallel to the long axis of the strain ellipsoid. This striation really seems to be a minute folding of the thin surface crust of the putty. It is parallel to the direction of slaty cleavage as developed in incompetent rocks sheared between competent beds. The above mentioned hybrid shear-tension fractures were oriented quite opposite to the slaty cleavage in natural layered rocks. The experiments thus support the theory that slaty cleavage develops parallel to maximum finite extensive strain in rocks rather than parallel to any of the shear directions.

Some of the putty batches used contain remnants of plasticene from previous experiments. These remnants were generally nebulous and contorted in irregular manner. During simple shear experiments some of these contorted nebulous remnants developed a kind of planar structure very similar to axial plane cleavage in slate.

Experiments with pure shear and various three dimensional strain.

It was found difficult to maintain homogenous strain under these types of deformation because of friction along the interfaces between yielding putty and the rigid plates which in most cases were used to transmit stresses. The resulting strain pattern of the embedded
plasticene sheets were therefore generally somewhat distorted by the nonhomogeneous component of the strain. A typical example of such distortion is seen in fig. D, pl. 5, which are results of simple compression between parallel plates. Because of friction between plates and putty, flow and longitudinal strain parallel to the plates are more intensive in the middle than along the margins. This gives rise to the outward bend in the plasticene sheets. The small folds are however due to the compressive strain. To reduce the effect of frictional drag, the plates were intermittently loosened from the putty body at short time intervals during the strain experiments.

In all experiments under this heading the plasticene strips or sheets were entirely enclosed in the putty and their final deformation studied in cuts through the putty cakes.

Results of various strain tests are shown in pls. 2, 5, 6. Explanation is given in the figure texts. Some of these strain features are worthy of special mention inasmuch as they did not develop in the simple shear tests.

Most striking is the result of uniform compression in the $x, y$-plane and corresponding extensive flow in $z$. Plasticene sheets oriented parallel to $x, y$ became buckled in a very complex fashion in response to such strain. Fig. B, pl. 6, shows an example of that. Cuts parallel to $z$ through such a buckled plasticene sheet will reveal rather ordinary ptygmatnic pattern, but cuts perpendicular to $z$ may show veins with closed outlines much like contour lines on a map of a mountainous terrain, fig. 10.

Under this type of strain two perpendicular veins both parallel to the $z$-axis will for example develop ptygmatnic folds simultaneously.

Compression parallel to $z$ and simultaneous uniform extension in the $x, y$-plane give rise to two-dimensional pinch-and-swell or boudinage structures of competent sheets oriented parallel to the $x, y$-plane.

Fig. A, pl. 5 shows the result of pure strain on putty with two enclosed plasticene sheets, one originally parallel to the $x, y$-plane and one originally almost parallel to $z$. The one sheet buckled into folds at the same time as the other ruptured to boudins. This demonstrates how, in natural rocks, ptygmatnic foldings and pinch-and-swell or boudinage structures may well form in response to one and the same period and geometry of strain in a rock complex.

Some of the figures, for example D, pl. 5, shows that wave length of the folds to some extent depends upon thickness of sheets.
Fig. 14: Deformation of original horizontal marker lines in a viscose or plastic body surrounding a stiff plate oriented parallel to the maximum compressive stress in pure shear. Compressive strain is about 33 %.

**Forces and mechanism of ptygmatic folding.**

The evolution of ptygmatic folding of veins can in some respect be compared with buckling of metal plates under compressive forces acting on the edges and parallel to the plate. Such buckling phenomena are of practical importance for mechanical engineers and have consequently been studied rather thoroughly within the elastic region of metals and other construction material. There are, however, several conditions which make the evolution of natural and experimental ptygmatic folds quite unlike the simple buckling tests of the engineers. For one thing the natural and experimental ptygmatic folding takes place largely within the plastic domain of the materials. Secondly the ptygmatically folded veins or plasticene strips are surrounded by material which is almost equally “firm” or viscose as the folded
bodies themselves. In practical buckling tests the viscosity of the surrounding media is negligible because the test pieces are generally surrounded by air or liquids. Thirdly the buckling forces are usually applied on the edges of the test sheets in the form of compressive pressures parallel to the sheets. This creates a uniform compressive plane stress throughout the entire sheet prior to buckling. In natural and experimental ptygmatic folding, however, stress in the sheet is chiefly caused by drag between the yielding host material and the competent sheet. This drag creates a non uniform plane stress which increases continuously from the edges toward the center or central cross section of the sheet. A brief analysis of this plane stress follows.

Consider a vein which is parallel to maximum compressive stress, \( \sigma_p \), in a homogeneous plastic rock of great extension in direction perpendicular to the vein, see fig. 14. Let the geometry of the strain be of pure shear type. We shall also assume that the host rock behaves like a very viscose Newtonian substance — i.e. a medium which has no yield point and in which rate of strain is proportional to applied stress. This assumption is reasonable for the slow creep of rocks at stresses below their elastic limit. At some distance away from the vein in \( x \) and \( z \) directions the host yields by simple compressive strain at constant rate parallel to \( z \). In the host rock outside the disturbing effect of the stiff vein there is no shear strain parallel to \( z \) which is the direction of principal compressive strain. As one approaches the competent vein, however, shear strain parallel to \( z \) becomes more and more pronounced, and in direct contact with the vein the rate of shear parallel to \( z \) reaches maximum values whereas the rate of compressive strain along \( z \) approaches zero provided that slip does not occur along the boundary. In a Newtonian substance rate of shear strain is proportional to shear stress. Consequently the surface of the vein is effected by a shear stress parallel to \( z \) and directed toward the central cross section \((x, y\text{-section})\) of the veins.

Because of symmetry the shear stresses on either side of the vein are identical in magnitude and sense at given distance \( z \) from the center. The relative magnitude of shear strain in the neighborhood of the vein is indicated in fig. 14 which shows how a set of originally even-spaced horizontal parallel marker lines have changed in the course of strain through an arbitrary time interval. The figure shows a case of about 33 % compression parallel to \( z \) outside the contact
Fig. 15: Deformation of a viscose or plastic body adjacent to a stiff plate. Only half of the plate, which is parallel to the $y$, $z$-plane, is shown (the black body along the $z$ axis). Origin of the coordinate system coincides with the center of the plate. Maximum compressive stress is parallel to $z$. The curves $z_0 - z'_1; z_0 - z'_2; z_0 - z'_3$ represent deformed version of the line $z_0 - z'_0$ after compressive strain parallel to $z$ in the host material outside the contact effect of the plate being $\varepsilon_z = 1/4, 1/2$ and 1. $\varepsilon_z$ is defined as $\frac{z_0 - z'_n}{z'_n}$.

Effect of the competent vein. Vein and host rock (plasticene and putty) are assumed to be completely welded together to prevent slip along the contact. The spacing between the marker lines is therefore unchanged at the vein contact. The drawing is based on actual strain tests with putty and plasticene sheets too strong to buckle, and especial care has been taken to reproduce the correct relative shear angles at the contact. It is found that the tangent of the shear angle $(\gamma = \frac{\partial z}{\partial x})$ is closely proportional to the distance $z$ from the center, a condition which is reasonable on theoretical grounds as well. Now the finite shear strain at every point along the contact developed in the course of the same time interval; we therefore conclude that the average rate of shear strain at a given contact point was proportional to the finite shear angle at that point. Since rate of shear strain is
proportional to shear stress in a Newtonian substance, it follows that shear stress along the vein surface is proportional to the distance from origin:

\[ \tau_z = c z \]

where \( c \) is a constant depending upon the viscosity of the host material and rate of compressive strain in the host outside the contact zone of the vein.

We shall now develop an approximation equation for shear strain at the ends of a thin, stiff vein (i.e. at \( z = z_0 \)) as a function of compressive strain parallel to the vein in the enclosing rock outside the contact zone. This will enable us to express the shear stress along the vein contact in terms of compressive stress on the host rock. Let the curve \( z_0 z' \) in fig. 15 represent the deformed version of the horizontal line \( z = z_0 \) after unit time interval, in which the strain rate has been constant. We shall assume that the stress in the \( z \) direction is constant \( = \sigma_z^* \) along the curve \( z_0 z' \). Consider now the column \( z'' = y \times dx \) where \( y \) is unity. This column has been compressed from \( z_0 \) to \( z'' \) in unit time. At the same time shearing has taken place along the sides of the column. Along the vertical side at \( x \) the shear varies from 0 at \( z = 0 \) to \( \frac{\partial z}{\partial x} \) at \( z = z'' \). Assuming that shear is a linear function of \( z \) the mean shear along this side is \( \frac{1}{2} \frac{\partial z}{\partial x} \). This shear has developed in unit time and therefore resists the compression of the column by a force \(-\frac{1}{2} \mu z \frac{\partial z}{\partial x}\) where \( \mu \) is the viscosity of the material.

Along the other side of the column at \( x + dx \) the shear varies from \( \frac{\partial (z + \frac{\partial z}{\partial x} dx)}{\partial x} \) at \( z = z'' \) to zero at \( z = 0 \). We shall again assume that the mean shear along this side is \( \frac{1}{2} \frac{\partial (z + \frac{\partial z}{\partial x} dx)}{\partial x} \). This shear helps to compress the column by a force \(-\frac{1}{2} \mu z \frac{\partial (z + \frac{\partial z}{\partial x} dx)}{\partial x}\).

* In the following discussion compressive stress and strain are considered positive.
The total compressive force effecting the column is then \( \sigma_z \partial x \frac{\partial (z + \frac{\partial z}{\partial x} \partial x)}{\partial x} \) which is balanced by the resistance force, \( \frac{1}{2} \mu z \frac{\partial^2 z}{\partial x^2} + 3 \mu \frac{z_0 - z}{z} \partial x \) where the last term represents resistance of a viscose body against simple compression. The constant \( \mu \) equals the viscosity coefficient since the strains have developed at constant rate through unit time interval.

Equating the compressive and the resistive forces gives:

\[
\frac{1}{2} z \mu \frac{\partial^2 z}{\partial x^2} = \sigma_z - \frac{z_0 - z}{z} 3 \mu, \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} = 2 \left( \frac{\sigma_z}{\mu} + 3 \right) z^{-1} - 6 z_0 z^{-2}
\]

of which the first derivative can be found by integration: (See Forsyth, 1943, p. 88)

\[
\frac{\partial z}{\partial x} = \pm \sqrt{4 \left( \frac{\sigma_z}{\mu} + 3 \right) \log \varepsilon + 12 z_0 z^{-1} + A}
\]

where \( A \) is a constant of integration.

According to the model fig. 15 we realize that both \( \frac{\partial^2 z}{\partial x^2} \) and \( \frac{\partial z}{\partial x} \) equal zero at \( x > x_c \), where \( x_c \) is the distance beyond which the contact effect of the vein vanishes. In the region \( x > x_c \), \( z = z' \) where \( z' \) is determined by the full compressive strain, \( \varepsilon_z = \frac{z_0 - z'}{z'} \), in the host rock. Since \( \frac{\partial^2 z}{\partial x^2} = 0 \) at \( z = z' \), eq. (35) gives:

\[
\frac{\sigma_z}{\mu} = 3 \frac{z_0 - z'}{z'} = 3 \varepsilon_z *
\]

for the region \( x > x_c \), or \( z = z' \).

The constant \( A \) is determining by setting \( \partial z/\partial x = 0 \) at \( z = z' \) in eq. (36):

\[
A = -12 (\varepsilon_z + 1) \log \varepsilon + \frac{z_0}{1 + \varepsilon_z} - 12 (\varepsilon_z + 1)
\]

* Note that according to our definition compressive strain has positive value.
in which the expressions $\frac{\sigma_z}{\mu} = 3 \epsilon_z$, and $z' = \frac{z_0}{\epsilon + 1}$ have been introduced. Eq. (36) then becomes:

$$40) \quad \frac{\partial z}{\partial x} = \pm \sqrt{12 (\epsilon + 1) \log_{\epsilon} \frac{(\epsilon + 1) z}{z_0} - (\epsilon + 1) 12 + 12 \frac{z_0}{z}}$$

Integration of this equation results in a family of deformation curves running from $z_0$ at the end of the vein to $z'_1$, $z'_2$, ..., $z'_n$ at $x > x_c$. Each value of the compressive strain, $\epsilon_z$, gives one member of this family of curves ending at $z'_1$, $z'_2$, etc. Geometrically integrated curves for $\epsilon_z = 1/4$, 1/2, and 1 are plotted in fig. 15.

It is interesting to see that the deformation curve from the ends of the vein for 33% compression (i.e. $\epsilon_z = 1/2$) in the putty-plasticene experiment fig. 14, coincides almost exactly with the theoretical curve for $\epsilon_z = 1/2$ (fig. 15). This is particularly true for the region close to the contact of the vein. That means that the shear-strain angle at the vein contact is rather accurately expressed by eq. (40).

It is worth noting that the contact of the competent vein — i.e. the region in which $\partial z/\partial x \leq 0$ — has a limited extension in the $x$-dimension. $x_c$ appears to equal $z_0$ rather closely. This is in good agreement with the experimental strain results as shown in fig. 14.

One of the most significant applications of eq. (40) lies in its ability to determine the contact shear strain, $\partial z/\partial x$, at the ends of the vein ($z = \pm z_0$, $x = 0$). This is achieved by setting $z = z_0$ at given $\epsilon_z$-values. Table I* gives correlated values for $\gamma_z = \partial z/\partial x$ at $z = z_0$ and $\epsilon_z = \frac{z_0 - z}{z}$ as determined by eq. (40). It is seen that the shear strain at the ends of the vein is very nearly proportional to the compressive strain parallel to $z$ in the host. This means that the rate of shear strain at the contact of the vein is proportional to the rate of compressive strain in the host at $x > x_c$.

The contact shear stress along the vein parallel to $z$ at the ends can now be expressed in terms of compressive stress, $\sigma_z$, in the host by the following relations: For rate of compressive strain in the host rock outside the contact zone: $\sigma_z = 3\mu \dot{\epsilon}_z$, where $\dot{\epsilon}_z$ is rate of strain, $\epsilon_z/\Delta t$. Hence: $\mu = \frac{\sigma_z}{3 \dot{\epsilon}_z}$.

* For Table I, see p. 151.
For shear strain at the contact at $z_0$:

41) \[ \tau_z = \mu \dot{\gamma}_0 \]

where $\dot{\gamma}_0$ is rate of shear strain, $\gamma \cdot /\Delta t$, and $\tau_z$ is shear stress, hence:

42) \[ \tau_z = \frac{\sigma_x}{3} \frac{\dot{\gamma}_0}{\varepsilon_z} \]

According to table I $\gamma /\varepsilon_z$ is almost constant for all $\varepsilon_z$. The mean value, 2.340, is inserted in eq. (42):

43) \[ \tau_z = 0.78 \sigma_x \]

We have for simplicity assumed $\sigma_x$ and $\sigma_y$ to be zero. This is very unlikely in rocks, hence $\sigma_x$ in equation (43) should be replaced by $\Delta \sigma_x$ which is the difference between $\sigma_x$ and $\sigma_z$:

44) \[ \tau_{x0} = 0.78 \Delta \sigma_x \]

It follows from eq. (40) that the contact shear-strain rate — and consequently the shear stress — at the ends of the vein is independent of the vein length, $2 z_0$, at given compressive-strain rate in the host.

The equations above do not show how $\tau_z$ varies along the vein contact, they only determine $\tau_{x0}$ at the terminals. However, we have concluded from experiments that the shear-strain rate, and consequently the shear stress, along a given sheet or vein is nearly proportional to distance from center. Since the shear stress at $z_0$ is $\tau_{x0} = 0.78 \Delta \sigma$ the contact shear stress at any point, $z$, is

45) \[ \tau_z = 0.78 \frac{\Delta \sigma_x}{z_0} z \]

where $z_0$ is the half-length of the vein.

Due to this drag then, the longitudinal compressive force parallel to $z$ across a $x$, $y$-section at distance $z$ from center in the vein is as reckoned per unit length in $y$:

46) \[
D = 2 \int_{z_0}^{z} \partial F = -2 \int_{z_0}^{z} \tau_z \partial z = -1.56 \frac{\Delta \sigma_x}{z_0} \int_{z_0}^{z} \partial z
\]

\[ = 0.78 \frac{\Delta \sigma_x}{z_0} (z_0^2 - z^2) \]

The factor 2 comes from the fact that equal stress acts on either side of the vein.
In addition to the drag-induced force there is pressure on the edges at $z = \pm z_0$ which results in an uniform compressive force throughout the entire length of the vein. Per unit length in $y$-dimension the edge force is:

$$F_e = \Delta \sigma_z' x_0$$

where $\Delta \sigma_z'$ is the difference $\sigma'_z - \sigma'_x$ in the host rock at the end of the vein, and $x_0$ is its thickness. Incidentally, one notes that $\sigma'_z$ is somewhat greater than the compressive stress along $z$ in the host outside the contact effect of the vein.

The total longitudinal compressive force across an $x$, $y$-section of the vein at $z$ is then per unit length in $y$:

$$F = F_e + D = \Delta \sigma_z' x_0 + 0.78 \frac{\Delta \sigma_z}{z_0} (z_0^2 - z^2)$$

This is the force which buckles the vein if the force exceeds the resistance against buckling as offered by the strength of the vein and the supporting effect of the enclosing host rock.

At some distance from the end points in a sufficiently long vein the compressive force can be much larger than the compressive stress in the host rock if it flows under the existing stress.

As an example assume $\Delta \sigma_z$ in host rock to be say 100 atm. Let a competent vein oriented parallel to maximum pressure be 400 cm long parallel to $z$ and 20 cm thick. At a section 100 cm from the ends, the compressive force parallel to $z$ in the vein per unit length in $y$ is then:

$$F = 100 \times 20 + 0.78 \frac{100}{200} (40,000 - 10,000) \text{ kg} = 13,700 \text{ kg},$$

or $13,700/20 \text{ atm.} = 685 \text{ atm.}$

If vein length is say 1000 cm ($z_0 = 500 \text{ cm}$) the compressive force 200 cm from the ends is:

$$F = 100 \times 20 + 0.78 \frac{100}{500} (250,000 - 90,000) \text{ kg} = 26,960 \text{ kg}$$

or $1,348 \text{ atm.}$

These figures show how pressure can accumulate considerably in competent bodies enclosed in incompetent rocks.

It is convenient to consider the resistance against buckling of an embedded vein in terms of two separate forces.
1. The strength against buckling which the vein possesses because of its elastical and rheological characteristics and its dimensions. This buckling strength of the vein at given P, T-conditions, is defined as the longitudinal compressive force needed to initiate buckling if the enclosing material offers no resistance whatever against the deformation — i.e. if the enclosing material was an ideal fluid without strength and viscosity. The effect of an ideal fluid of this type would only be to keep the vein under a given confining pressure thereby partly controlling the mechanical characteristics of the vein material.

2. The resistance against buckling of the vein as offered solely by the strength and viscosity of the enclosing medium. Sidewise motion of folds into the host material is obviously resisted by the viscous or plastic host.

Let us first consider the buckling strengt of the vein. A plate of elastic material effected on opposite edges by a compressive longitudinal force equal to or greater than its buckling strength, is mechanically unstable in the sense that a very minute transversal force is sufficient to initiate buckling which will not pop back upon withdrawal of the small transversal force. If the plate is unable to yield plastically, the buckled state of the compressed plate is stable unless the compressive force is considerably larger than the buckling strength in which case the plate will collapse. If plastic creep occurs in the plate, however, the amplitude of the buckles will increase with time even if the applied stress does not exceed the buckling strength.

Since the theory of elastic buckling of plates is well understood we shall firstly consider that and then see how it may be modified to account for plastic buckling of the type occurring in ptygmatic veins. Fig. 16 shows a plate which is elastically bend by a force $P$ acting parallel to $z$. In order that static stability shall exist the moment of the buckling force, $P \cdot A \Delta x$ must be balanced by the moment of the stresses in the plate. If $P$ is appreciably greater than this critical value the plate will continue to bend until collapse; if $P$ is smaller, the plate will not bend at all.

The plate is compressed on the concave side of the neutral central surface of no strain, and extended on the convex side of that surface. Because of symmetry the tensile stress on the convex side equals numerically the compressive stress on the concave side at given distance from the central surface. Let the longitudinal stress parallel
to the plate be $\pm \sigma$ at distance $\pm x$ from the center surface. The moment of the tensile and compressive stresses is then:

$$M = 2 \int_0^h \sigma x \, dx$$

$\sigma$ is zero at the central neutral plane and reaches numerically maximum values at $x = \pm h$. The change of $\sigma$ with $x$ in the region $-h, +h$, is linear:
\[ \sigma = \frac{\sigma_h}{h} x \]

where \( x \) varies within the limit \( \pm h \). \( \sigma_h \) is the stress at the surfaces at \( x = \pm h \). The value of \( \sigma_h \) is determined by the magnitude of longitudinal strain at \( x = \pm h \). Provided that the half-wave constitutes a circle arc, this strain is:

\[ \epsilon_h = \pm \frac{h}{R} \]

where \( R \) is the radius of curvature of the buckle. Since the strain is elastic, the ratio between stress and strain equals Young’s nodulus, \( E \):*

\[ \sigma_h = E \cdot \epsilon_h = E \frac{h}{R} \]

For folds with small \( \Delta x/R \) ratio as we are considering here the following approximate relations can be used:

\[ \frac{2 \Delta x}{\lambda} = \frac{1}{4} \frac{\lambda}{R}, \text{ or } R = \frac{1}{32} \frac{\lambda^2}{\Delta x} \]

where \( \lambda \) is the wave length of the folds. Eq. (52) then becomes:

\[ \sigma_h = E h \frac{32 \Delta x}{\lambda^2} \]

The moment due to elastic stress is consequently:

\[ M = 2 \int_0^h E \frac{32 \Delta x}{\lambda^2} x^2 \, dx = 21.3 \frac{\Delta x}{\lambda^2} h^3 E \]

Equating the moment of the applied external force, \( P \cdot \Delta x \), with elastic moment, \( M \), gives:

\[ P = \frac{21.3 E h^3}{\lambda^2} \]

\( P \) is the force, per unit length in \( y \)-dimension, acting in the plane of the plate parallel to \( z \) which is required to make the plate of thickness \( 2h \) buckle elastically in folds with wave length \( \lambda \).

* For simplicity we shall disregard strain parallel to the fold axis of the buckle. I.e. Poisson’s ratio equals zero.
More sophisticated evaluations of buckling strength of plates show that the Poisson's ratio and the ratio between the side dimensions of the plate are of significance as for example discussed by J. J. Staker, 1941. A comparison of eq. (56) with the more elaborate buckling expressions shows that equation (56) is valid for any number of buckles in plates very elongate in the \( y \)-dimension and for square plates if more than 3 buckles are forced to form.

If the plate is not restrained by any outside forces except along the two opposite edges on which pressure is applied, minimum buckling force is attained when the plate makes one buckle or one half-wave of which the inflection lines coincides with the edges where force is applied. This gives maximum value for wave length in eq. (56) and consequently minimum value for \( P \). If sidewise "popping" is hindered along equal-spaced lines parallel to the compressed edges however, the plate must make several half-waves if it is to buckle at all. As shown by eq. (56) the force necessary to cause multibuckling of this kind increases rapidly with decreasing wave length.

It is interesting to check the applicability of the elastic buckling equation on rocks. For granitic rocks, for example, \( E \) is of the order \( 5 \times 10^{11} \) dynes/cm\(^2\) ("Handbook", 1942 p. 79–80). \( k = 1 \) cm, and \( \lambda = 20 \) cm are reasonable dimensions for an average ptygmatic granitic veinlet. According to these values the buckling force becomes:

\[
P = \frac{21.3.5.10^{11}}{400} \approx \frac{1}{4} 10^{11}
\]

dynes per unit length in \( y \)-dimension. Since thickness is \( 2 h \), this means a compressive longitudinal stress along the vein equal to \( \frac{1}{8} 10^5 \) bars.

This seems an unreasonable high figure, particularly as it must be considered not as absolute stress, but rather as stress difference between the \( z \)- and \( x \)-and/or \( y \)-dimensions.

Let us then attempt to adapt the considerations above to buckling by a slow plastic creep in the plate of the kind likely to take place in veins or other sheet-shaped rock bodies being folded.

The crucial difference between such a mechanism and the elastic case is that in plastic creep rate of strain, \( \frac{\partial \epsilon}{\partial t} = \dot{\epsilon} \), rather than strain itself is proportional to stress. (For simplicity we assume again
that the rocks behave as very viscose Newtonian substances.) It is therefore conceivable that forces much less than the elastic buckling load are capable of causing buckling by a plastic creep mechanism.

Let fig. 16 represent a plastic plate with very small curvature. As in the elastic model the moment due to the external force must be balanced by the stress moment of the slightly buckled plate. Consequently the condition:

\[ P \Delta x = M = 2 \int_0^h \sigma x \, dx, \text{ or } 2 \int_0^h \frac{\sigma_h}{h} x^2 \, dx = \frac{2}{3} \sigma_h h^2 \]

is still valid. However, rather than causing a static elastic strain the longitudinal stress now gives rise to a slow plastic creep on either side of the neutral middle surface of the fold. The relationship between compressive, respective tensile, stress and creep in a Newtonian substance is:*

\[ \sigma_h = 3 \mu \frac{\partial \varepsilon_h}{\partial t} \]

where \( \mu \) is viscosity for plastic creep and \( \frac{\partial \varepsilon_h}{\partial t} \) is rate of compressive or extension strain in the surface layers of the folded plate. According to eqs. (51) and (53):

\[ \varepsilon_h = \frac{32 h \Delta x}{\lambda^2} \text{ (approximately)} \]

For our purpose it is most convenient to relate strain in the surface layers of the plate, \( \varepsilon_h \), to relative shortening along the \( z \)-dimension of a buckle, \( \varepsilon_z \), as caused by the bending. We shall only concern ourselves in the following with the initiation of small-amplitude buckles in a plate. For such initial waves with very small \( \Delta x/\lambda \) ratio, it can be shown geometrically, (fig. 16) that the decrement in wave length, \( \Delta \lambda \), as caused by bending, is related to amplitude and wave length as approximately follows:

\[ \frac{1}{4} \frac{\Delta \lambda}{\Delta x} = \frac{\Delta x}{\frac{1}{4} \lambda}. \]

Relative shortening along \( z \) for initial waves is defined as

* This relation holds for "pure shear".
61) \[ \varepsilon_z = \frac{\text{length of arc} - \lambda}{\lambda} = \frac{\Delta \lambda}{\lambda} \],

hence:

62) \[ \Delta x = \frac{1}{4} \lambda \frac{1}{\varepsilon_z^2}, \]

which inserted in eq. (59) gives:

63) \[ \varepsilon_k = 8 \frac{h \frac{1}{\varepsilon_z^2}}{\lambda}. \]

Partial differentiation with respect to \( \varepsilon_z \) and \( \lambda \) results in:

64) \[ \frac{d \varepsilon_k}{\lambda} = \frac{4 h}{\lambda} \left( \varepsilon_z^{-\frac{1}{2}} \frac{\partial}{\partial \varepsilon_z} - 2 \varepsilon_z^{-\frac{3}{2}} \frac{\partial \lambda}{\partial \varepsilon_z} \right) \]

where \( \frac{\partial \lambda}{\partial \varepsilon_z} \) can be replaced by \( \partial \varepsilon_z \).

Accordingly:

65) \[ \sigma_k = \frac{12 \mu h}{\lambda} \left( \varepsilon_z^{-\frac{1}{2}} - 2 \varepsilon_z^{-\frac{3}{2}} \right) \frac{\partial \varepsilon_z}{\partial t} \]

and:

66) \[ P \Delta x = \frac{8 \mu h^3}{\lambda} \left( \varepsilon_z^{-\frac{1}{2}} - 2 \varepsilon_z^{-\frac{3}{2}} \right) \frac{\partial \varepsilon_z}{\partial t} \]

Replacing \( \Delta x \) by \( \frac{1}{4} \lambda \varepsilon_z^{-\frac{1}{2}} \) gives the longitudinal buckling force:

71) \[ P = \frac{32 \mu h^3}{\lambda^2} \left( \varepsilon_z^{-1} - 2 \right) \frac{\partial \varepsilon_z}{\partial t}, \]

72) \[ P = \frac{32 \mu h^3}{\lambda^3} \left( \frac{\lambda}{\Delta \lambda} - 2 \right) \frac{\partial \lambda}{\partial t}. \]

\( P \) is to be interpreted now as the force parallel to \( z \), reckoned per unit length in \( y \), which gives a rate of buckling or diminishing wave length equal to \( \frac{\partial \varepsilon_z}{\partial t} = \left( \frac{\partial \lambda}{\lambda} \right) \frac{\partial \lambda}{\partial t} \). The equation is only valid for initial waves with very small curvatures; that is to say the quantity, \( \varepsilon_z \) which represents the relative shortening of the wave length from the...  

* Compressive strain according to this definition assumes positive values.
plane stage to the slightly buckled stage, is small, say less than 0.01. As one should expect, the force increases with rate of buckling. It is also in accord with expectations that \( P \) decreases with increasing magnitude of buckling (as expressed by \( \varepsilon^{-1} \)) because of the increased moment of the force.

It is worth noting that there is no lower limit to the buckling force for plastic buckling within the creep domain below the yield point. If \( \frac{\partial \varepsilon_z}{\partial t} \) approaches zero, \( P \) also approaches zero provided the 
\( \varepsilon_z \leq 0 \). The condition \( \varepsilon_z \leq 0 \) means that an ideal homogeneous plate cannot start buckling if it is mathematically straight and the longitudinal compressive forces are symmetric with respect to the central plane. A minute initial bend is necessary to give moment to the longitudinal force. The requirement \( \varepsilon_z \leq 0 \) is an expression of this condition for \( \varepsilon_z \), which equals the difference between arc length and wave length divided by wave length, vanishes only for mathematically straight plates. Of course, under natural and experimental conditions the materials are not flawless either in structure or dimensions. There will always be smaller or larger initial curvatures along veins, points of weakness, and/or unsymmetric application of forces to give bending moment to the existing stresses.

We concluded that the forces necessary to buckle granitic veins elastically are much higher than reasonable for crustal stresses. Buckling by plastic creep, however, gives a more likely picture. Let \( h \) be 1 cm and \( \lambda \) be 20 cm as in the elastic example. \( \mu \) having any values between say \( 10^{14} \) and \( 10^{21} \) poises is possible for solid rocks. As an example a value of \( 10^{18} \) poises will be used. Assume then that the original curvature of the vein corresponds to \( \varepsilon_z = -0.01 \), or that 
\[
\frac{\Delta x}{\lambda} = 0.025 \quad (\text{See eq. (62) p. 144}).
\]
For \( P \) a rather small value of say 100 kg or \( 10^8 \) dynes will be chosen. Inserting these values in eq. (71) gives rate of buckling of the fold as expressed in terms of \( \frac{\partial \varepsilon_z}{\partial t} \) or \( \frac{\partial \lambda}{\partial t} \):

\[
10^8 = \frac{32 \cdot 10^{18} \cdot 1}{400} (100 - 2) \frac{\partial \varepsilon_z}{\partial t},
\]

or

\[
\frac{\partial \varepsilon_z}{\partial t} = \frac{\partial \lambda}{\lambda \partial t} \sim 10^{-11} \text{ sec}^{-1} \sim 0.03
\]

per hundred years. In other words, the wave length of 20 cm be-
comes shortened by some 3 per cent of its original length or 0.6 cm in the course of hundred years. This rough calculation shows that the equation for plastic buckling below the yield point gives reasonable results contrary to the elastic buckling model.

We shall now consider the resistance against initial folding of the vein as offered by the enclosing material because of its firmness and viscosity.

It has unfortunately not been possible to find rigorous treatment of a parallel problem in literature, but a simple analysis shows that the resistance against initial buckling of a thin sheet offered by an enclosing viscose or plastic medium increases with increasing wave length of the folded sheet. As a matter of fact the enclosing rock tends to force a stressed vein to develope an infinite number of fininitesimal folds whereas the strength of the vein itself tends to produce one single half-wave covering the entire vein length. The actual arc length of ptygmatic folds are results of a compromise between these two opposite tendencies.

In analogy with the equation for slow transversal motion of long cylinders or elongate plates in a highly viscose medium, as for example studied by Lamb (1932, p. 616), we shall assume that a relationship of the following kind is applicable to initiation and growth of folds of a plate enclosed in a viscose or plastic substance with viscosity $\mu$:

$$F_x = C \mu \frac{\partial (Ax)}{\partial t}$$

where $F_x$ is the force in $x$-direction, reckoned per unit length along the fold axis, that is needed to increase the amplitude of one half-wave at a rate $\frac{\partial (Ax)}{\partial t}$. $F_x$ is thus the resistance force connected with the deformation of the host material. Now the buckling force acting on veins as well as their buckling strength have been considered in terms of components in the $z$ directions along the veins. To make $F_x$ comparable with these forces, then, it must be transformed into an equivalent resistance force, $F_z$, acting in the $z$-direction along the vein. The transformation is readily done by the law of equal moments:

$$F_z Ax = M,$$
where $M$ is the moment of the resistance force in $x$-direction acting on $\frac{1}{4}$ wave.

Assuming that $F_x$ is uniformly distributed over the whole half-wave, the average force per unit area of the wave is $\frac{F_x}{2\lambda}$, per unit length along $y$, and the moment is:

\[
M = \frac{1}{4} \int_0^{\lambda} F_x z \, dz = \frac{1}{4} \lambda \int_0^{\lambda} 2C \frac{\mu}{\lambda} \frac{\partial (Ax)}{\partial t} z \, dz = \frac{1}{16} C \mu \frac{\partial (Ax)}{\partial t} \lambda.
\]

Consequently:

\[
F_x = \frac{1}{16} C \mu \frac{\partial (Ax)}{\partial t} \lambda.
\]

Replacing $\partial (Ax)$ and $Ax$ according to eq. (62) and its derivative gives $F_z$ as a function of relative shortening of the wave length, $\varepsilon_x$, and rate of shortening, $\partial \varepsilon_x/\partial t$:

\[
F_z = \frac{1}{32} C \mu \lambda (\varepsilon_x^{-1} + 2) \frac{\partial \varepsilon_x}{\partial t},
\]
or:

\[
F_z = \frac{1}{32} C \mu \left( \frac{\lambda}{Ax} + 2 \right) \frac{\partial \lambda}{\partial t}.
\]

The most significant information contained in this equation is that the resistance force against buckling increases with increasing wave length and increasing viscosity of the host material for given relative shortening and given rate of relative shortening. The effect of the enclosing material is consequently to make the wave length as small as possible. If the vein had no strength itself, the host rock would make the wave length infinitly small. This means that the vein would only be compressed along $z$ and thickened parallel to $x$, but not folded. That is what happens in the experiments when vein and host have the same viscosity and strength.

The total longitudinal compressive force, as reckoned per unit axial length, necessary to make the vein buckle at a rate $\partial \varepsilon_x/\partial t$, or $\frac{\partial \lambda}{\lambda \partial t}$ is then:
\[ F = F_e + P - \frac{1}{32} C \mu_1 \lambda (\varepsilon_z^{-1} + 2) \frac{\partial \varepsilon_z}{\partial t} + 32 \frac{\mu_2 h^3}{\lambda^2} (\varepsilon_z^{-1} - 2) \frac{\partial \varepsilon_z}{\partial t} \]

where the first term represents the resistance offered by the enclosing rock due to its viscosity, and the second term represents the buckling resistance of the vein itself.

Eq. (75) is only valid for folds with small \( \Delta x/\lambda \) ratio. In other words the quantity \( \varepsilon_z \) must be small, say less than 0.01.

If everything except wave length and force is kept constant in eq. (75) it appears that \( F \) has a minimum value for a certain wave length. This is the very wave length which will develop when buckling starts. Of course, as buckling continues in response to compressive strain in the host rock, the wave length shortens. The length of arc of the waves, however, remains constant and equal to the initial wave length during the subsequent shortening (provided that the length of the vein does not change during the deformation). The magnitude of the initial wave length can therefore always be determined in the field even on mature ptygmatic folds.

An expression for the initial wave length, \( \lambda_i \), is readily determined by differentiating eq. (75) with respect to \( \lambda \) and equate to zero:

\[ dF = \frac{1}{32} C \mu_1 (\varepsilon_z^{-1} + 2) \frac{\partial \varepsilon_z}{\partial t} d\lambda - 64 \frac{\mu_2 h^3}{\lambda^2} (\varepsilon_z^{-1} - 2) \frac{\partial \varepsilon_z}{\partial t} \frac{\partial \lambda}{\lambda^3} = 0 \]

Hence:

\[ \lambda_i = \frac{3}{32} \frac{C}{\mu_1} \frac{2048}{\mu_2} \frac{\varepsilon_z^{-1} - 2}{(\varepsilon_z^{-1} + 2)} \]

or

\[ \lambda_i = \sqrt[3]{\frac{2048}{C} \frac{\mu_2}{\mu_1}} \]

because \( \varepsilon_z \) is necessarily very small when buckling starts. This expression for initial wave length is in harmony with general field experience. It is generally true that \( \lambda_i \) increases with thickness of the vein and with increasing ratio of viscosity (strength) of vein to viscosity (strength) of host rock.

Introducing \( \lambda_i \) in eq. (75) gives the minimum force necessary to buckle the vein:
where

\[ F_{\text{min}} \approx \left[ \frac{1}{32} C \frac{\mu_2}{\mu_1} \sqrt{n} + 32 \frac{\mu_2}{\mu_1} \frac{h}{\eta} \frac{\partial \varepsilon_z}{\partial t} \right] \]

It is interesting to note that the initial wave length, \( \lambda_i \), does not depend upon the rate of buckling, \( \partial \varepsilon_z / \partial t \), or the magnitude of relative shortening, \( \varepsilon_z \), at which buckling starts. However, the minimum force at which the initial development is indeed strongly influenced by both rate of compression and absolute magnitude of compression. This condition is significant for the evolution of ptygmatic folds in rocks because if initial wave length — which is identical to the length of arc of the mature folds — was significantly dependent upon rate of compressive strain in rocks it should probably result in very erratic correlation between thickness and arc length of natural ptygmatic veins. The fact that \( \lambda_i \) is practically independent of relative shortening (if it is small) along \( z \), as expressed by \( \varepsilon_z \), at the moment buckling starts indicates that initial wave length is not affected by original small-amplitude bends along an uniform vein. The term \( \varepsilon_z \) in the equations is an expression of the curvature of such a small bend; see eq. (61) p. 144.

It is unfortunate that the factor \( C \) in eq. (67) is unknown because otherwise measurements of length of arc and thickness on ptygmatic folds — which according to the model above are initiated at the minimum compressive force \( F_{\text{min}} \) — could be introduced in the expression for \( \lambda_i \) (eq. (78)) and thus enable us to determine relative creep viscosities for vein and host rock. It is hoped that future studies will give a more accurate equation for the buckling resistance due to host-rock deformation adjacent to folds.

We are now in position to follow in some detail the evolution of a ptygmatic vein which is oriented parallel to maximum compressive strain in an incompetent rock undergoing pure-shear deformation. Drag along the contact induces a longitudinal compressive force which increases from the ends to the central section of the vein according to eq. (48). This force tends to make the vein buckle. The rate of buckling must be very slow along segments of the vein near its ends where the force is small. However, at a certain distance,
$\Delta z = z_0 - z_c$ from the ends the accumulated compressive force reaches a certain critical value. This critical value of the force is characterized by being exactly sufficient to make the vein buckle at a rate $\frac{\partial \lambda}{\lambda \partial t}$, or $\frac{\partial \varepsilon_z}{\partial t}$ equal to the rate of compression in the host rock outside the contact zone of the vein. The result of this is then that folds with wave length, $\lambda$, determined by eq. (78) are initiated rather suddenly and simultaneously over the central vein segment from $- z_c$ to $+ z_c$. During further evolution the folds in the central segment continue to shorten in pace with the host-rock compression outside the contact zone. In other words $\frac{\partial \varepsilon_z}{\partial t}$ or $\frac{\partial \lambda}{\lambda \partial t} = \frac{\partial \varepsilon_c}{\partial t}$ where $\varepsilon_c$ is compressive strain in host rock $\sigma$ outside the contact zone. Shortening of folds in the central segment continues in pace with the distant host-rock compression, $\frac{\partial \varepsilon_c}{\partial t}$, for some time because the buckling resistance decreases with increasing amplitude of the folds. As the folds become thightened, however, the resistance against buckling increases mainly because of pinching of host-rock material in the concave portions of the folds. Hence it seems likely that further shortening of the folds in the central segment goes on more slowly than the distant host-rock shortening.

Folding of the end segments of the vein also occurs but at a slower rate because of less compressive force. (Note the straight undisturbed end segments in many of the experimental ptygmic structures. See pls. 1 to 6).

If the vein is evenly thick and both vein and host rock are homogeneous the initial wave length should not be influenced by force of compression or rate of deformation, as shown by eq. (78). and uniform wave length or rather arc length along the entire vein should be the result.
TABLE I

Relations between $\epsilon_z = \frac{z_0 - z'}{z'}$ and $\gamma_0 = \left( \frac{\partial z}{\partial x} \right)_{z_0}$ according to expression:

$$\left( \frac{\partial z}{\partial x} \right)_{z_0} = \pm \sqrt{\frac{12}{\epsilon_1}} \log_\epsilon (\epsilon + 1) - 12 (\epsilon + 1) + 12$$

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ACKNOWLEDGMENT

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**PLATE TEXT**

**Pl. 1:** *Simple shear deformation of putty body with embedded plasticene strips.*

A, B, C, and D, represent consecutive steps of a deformation at which the upper wooden block has been moved toward the left. The folded plasticene strip to the right lies in the compression direction. Note that folding starts in the middle of the strip where longitudinal compressive strain is highest. The strip to the left is parallel to one of the two principal shear planes and has neither been folded nor stretched. Hybrid shear-tension fractures are seen in figs. C and D.

**Pl. 2:** *Deformation of putty bodies with embedded plasticene strips.*

A, B, and C represent consecutive steps on simple shear deformation at which the upper wooden block has been slid toward the left. The plasticene strip to the left is effected by a component of extensive strain and has consequently been stretched with break in the middle where tensile stress is at a maximum. Note the unfolded end segments of the compressed strip to the right. Hybrid shear-tension fractures are visible in fig. C. Fig. D is the result of pure shear deformation (vertical compression) of a sheet of plasticene embedded in putty. Several cuts of a single compressed cake are shown.

**Pl. 3 and 4:** *Deformation of putty body with embedded plasticene strips.*

A, B, C, D, E, and F represent consecutive steps on a continuous deformation at which the upper block has been slid toward the left. G shows reversal of the movement after step F. Both strips are parallel to a component of compressive strain, the left strip, however, being a little thicker than the right one. (This difference in thickness may not show up on the photos because the surface of the strips were somewhat smeared during smoothening of the putty prior to deformation). Note the difference in wave length of the two strips, and that folding starts in the central segment of the strips. Fig. H pl. 4 shows different wave length of three buckled plasticene strips with unlike thickness. The thickness of the strips increases from right to left.
Pl. 5: **Deformation of plasticene strips embedded in putty.**

*A* shows cuts of a cake effected by compression in vertical direction. The black pinching-and-swelling body in the middle was a uniformly thick plasticene layer oriented perpendicular to maximum compression. The gray folded inclined “vein” in the center was a straight plasticene sheet oriented somewhat inclined to compression axis, but yet having a component of compressive strain parallel to itself.

*B* shows deformation of a plasticene strip and originally parallel vertical marker lines during simple shear.

In *C* a folded plasticene strip and striations parallel to the long axis of the strain ellipsoid are shown.

*D* shows cuts of a vertically compressed putty cake with two vertical plasticene sheets. Note the difference in wave length as related to different thickness.

Pl. 6: **Deformations of putty with embedded plasticene sheets.**

*A*: Cuts of vertically compressed putty cake with ptygmatic plasticene sheet.

*B*: Result of uniform compression in the plane of the picture and elongation perpendicular to the picture. It shows the folded surface of an originally plane plasticene sheet which was embedded in putty and oriented in the plane of compression. The putty was removed from one side of the crumpled plasticene sheet after deformation.

*C*: Plasticene strip in putty deformed in simple shear.

*D*: Two plasticene sheets originally completely embedded in putty deformed by simple shear at which the upper wooden block was slid toward the left. The two sheets plunge down at about $30^\circ$ toward the middle of the photographs as indicated by the arrows on the fold axes. The putty which covered the folded sheets has been removed after the deformation run was completed.